Random number generator

Problem 0: *Random or not?*

1. Prepare a qubit in a superposition state using a single Hadamard gate. Measurement the outcome and show that a quantum measurement yields probabilistic results

   Hint: Run the circuit with 1 shot and repeat the measurement 10 times. What do you observe? Run a measurement with 1024 shots.

2. Apply a second Hadamard after the first one and measure. Is the result random?

Decoherence

Problem 1: *Relaxation and coherence*

Real quantum computers must deal with decoherence, or the loss of information due to environmental disturbances (noise).

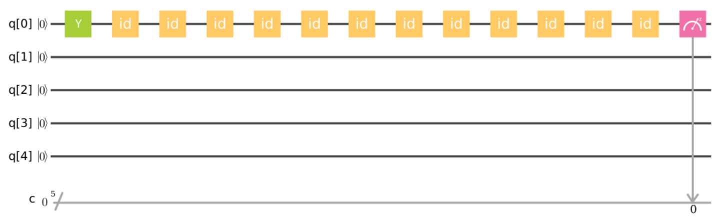
1. Energy relaxation and T_1

   One important decoherence process is called energy relaxation, where the excited state $|1\rangle$ decays toward the ground state $|0\rangle$. The time constant that describes this process, T_1, is an extremely important figure-of-merit for any implementation of quantum computing.

   (a) Prepare a qubit in the excited state $|1\rangle$ and measure the occupation of the excited state as you vary the time delay between preparation and measurement. Use the identity gate to add a time delay to the circuit (the identity gate does nothing but wait for the time it takes to perform a single gate operation). Determine T_1 in units of the single gate time.

   Hint: The maximum number of gates per qubit is limited to 40 and thus the maximum wait time. Perform your measurements on the qubit with the shortest T_1.

   Solutions:

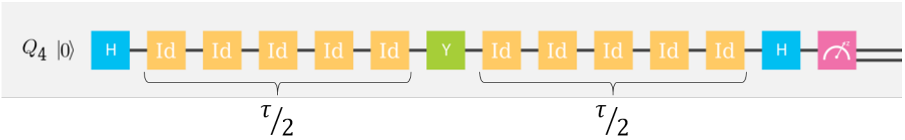
   (b) Check the relaxation time stated for the specific qubit and determine the time it takes to perform a single gate operation.

2. Dephasing and T_2

   Dephasing is another decoherence process, and unlike energy relaxation, it affects only superposition states. It can be understood solely in a quantum setting as it has no classical analog. The time constant includes the effect of dephasing as well as energy relaxation, and is another crucial figure-of-merit.

   (a) Explain the circuit shown below by describing the trajectory of the Bloch vector on the Bloch sphere. Which time constant is measured with this circuit?

   (b) Determine the time constant and discuss your result.

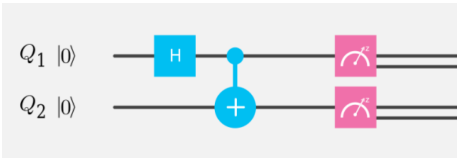


Bell states and teleportation

In this section you will perform a quantum teleportation of a single qubit from Alice to Bob. As an introduction, please have a look at the paper from the Wallraff group at ETH published in Nature (doi:10.1038/nature12422), especially the introduction and Fig. 1(a), as well as at the Wikipedia article on teleportation (https://en.wikipedia.org/wiki/Quantum_teleportation).

Problem 2: *Create a Bell state*

The two-qubit state $|00\rangle + |11\rangle$ is one of four possible maximally entangled Bell states. In a quantum computer, it can be created from a $|00\rangle$ input state using a CNOT and a Hadamard gate, as shown in the following Figure.



1. Show mathematically that $|00\rangle$ is transformed into the Bell state $|00\rangle + |11\rangle$.

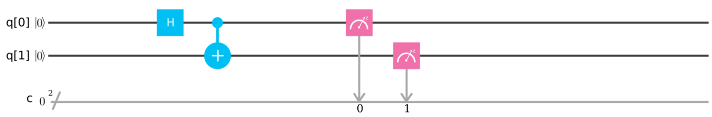
   Solutions:

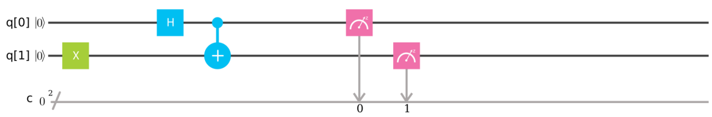
$$|00\rangle \xrightarrow{H} |00\rangle + |10\rangle \xrightarrow{CNOT} |00\rangle + |11\rangle \tag{1}$$

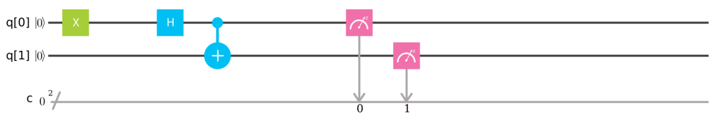
2. Use the Quantum Experience and create the Bell state with the gate sequence mentioned above. Measure the outcome. Start with using the 'realistic quantum processor' simulation. Try out the real quantum computer with 1024 runs. What do you observe if you use only one run? Explain the result.

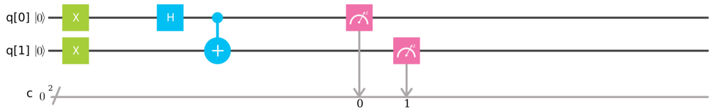
3. What happens if you use $|01\rangle$, $|10\rangle$ or $|11\rangle$ as input states? Implement the circuit in Quantum Experience and measure the outcome.

   Solutions:









4. Why do $|00\rangle$ and $|10\rangle$ give the same outcome? Which states do you expect mathematically? How could one differentiate between the two outcomes using the quantum computer?

   Solutions:
   Mathematically one expects the Bell states

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \tag{2}$$

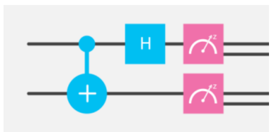
$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \tag{3}$$

   However, the projection of $|\Phi_+\rangle$ and $|\Psi_+\rangle$ onto the basis $\{|0\rangle, |1\rangle\}$ gives the same probabilities. We need to transform the Bell states into the states spanned by the basis set of $|0\rangle$ and $|1\rangle$ and then perform a projective measurement as explained in the next part.

Problem 3: *Bell state measurement*

In a quantum teleportation, a qubit C in a state $|\psi\rangle$ is teleported from Alice to Bob. As you can read in the Wikipedia article mentioned above, for this to be possible, Alice and Bob need to share an entangled pair of qubits A (located with Alice) and B (located with Bob). The essential part of the teleportation protocol is that Alice performs a Bell state measurement on her qubit pair A and C. Depending on the outcome of this measurement (there are four possibilities), Qubit B at Bob's location collapses into one of four states that are either identical or very similar to the original state of qubit C.

1. The quantum computer can perform a projective measurement of the qubit component along the Z direction, i.e. it can determine whether a qubit is in a $|0\rangle$ or in a $|1\rangle$ state. It can do that on many qubits simultaneously. In order to perform a Bell state measurement, we need to transform the Bell states into the states spanned by the basis set of $|0\rangle$ and $|1\rangle$ and then perform a projective measurement. Show mathematically that the combination of CNOT and Hadamard gates as seen in the Figure below maps the four possible Bell states into the measurement states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$.

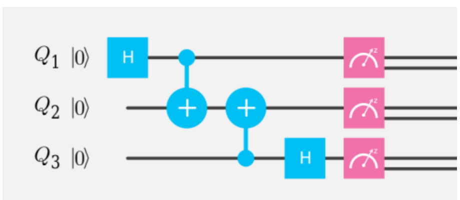


2. Program the quantum computer to create any of the four Bell states, and then do a Bell measurement.

Problem 4: *Quantum teleportation*

Now we have everything together to perform a quantum teleportation.

1. Do a quantum teleportation of a $|0\rangle$ state by implementing the protocol discussed in the Wikipedia article and implemented as in the following Figure (where Q3 is teleported to Q1, Alice possesses qubits Q2 and Q3, Bob Q1).



2. For each possible Bell state of the pair Q2 and Q3, we obtain a probability density of the teleported qubit Q1 being in a $|0\rangle$ or in a $|1\rangle$ state. Validate experimentally whether this probability density reflects the state of Q3 originally prepared by Alice. For this, prepare different initial states of qubit Q3 (read the Quantum Experience User Guide on how to do that) and create a table with the probability densities of the measured outcome.

   Solutions: For the Bell state $|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ we get

$$|000\rangle \tag{4}$$
$$\xrightarrow{\texttt{H(Q1)}} \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle) \tag{5}$$
$$\xrightarrow{\texttt{CNOT(Q2cQ1)}} \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) \tag{6}$$
$$\xrightarrow{\texttt{CNOT(Q2cQ3)}} \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) \tag{7}$$
$$\xrightarrow{\texttt{H(Q3)}} \frac{1}{2}(|000\rangle + |110\rangle + |001\rangle + |111\rangle) \tag{8}$$

$$|001\rangle \tag{9}$$
$$\xrightarrow{\texttt{H(Q1)}} \frac{1}{\sqrt{2}}(|001\rangle + |101\rangle) \tag{10}$$
$$\xrightarrow{\texttt{CNOT(Q2cQ1)}} \frac{1}{\sqrt{2}}(|001\rangle + |111\rangle) \tag{11}$$
$$\xrightarrow{\texttt{CNOT(Q2cQ3)}} \frac{1}{\sqrt{2}}(|011\rangle + |101\rangle) \tag{12}$$
$$\xrightarrow{\texttt{H(Q3)}} \frac{1}{2}(|011\rangle + |101\rangle + |010\rangle + |100\rangle) \tag{13}$$

   This works similary for all other Bell states

3. Using the Quantum Experience, show experimentally that if you create qubit Q3 in a $|X\rangle$ or $|Y\rangle$ state (or any state on the equator of the Bloch sphere), you will measure an equal probability of 1/8 for the 8 possible outcomes. Show formally why this is the case.

   Solutions:

$$|000\rangle \tag{14}$$
$$\xrightarrow{\texttt{H(Q3)}} \frac{1}{\sqrt{2}}(|000\rangle + |001\rangle) \tag{15}$$
$$\xrightarrow{\texttt{H(Q1)}} \frac{1}{2}(|000\rangle + |001\rangle + |100\rangle + |101\rangle) \tag{16}$$
$$\xrightarrow{\texttt{CNOT(Q2cQ1)}} \frac{1}{2}(|000\rangle + |001\rangle + |110\rangle + |111\rangle) \tag{17}$$
$$\xrightarrow{\texttt{CNOT(Q2cQ3)}} \frac{1}{2}(|000\rangle + |011\rangle + |110\rangle + |101\rangle) \tag{18}$$
$$\xrightarrow{\texttt{H(Q3)}} \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |011\rangle + |010\rangle + |110\rangle + |111\rangle + |101\rangle + |100\rangle) \tag{19}$$

Constructing two qubit gates

Problem 5: *Superdense coding*

Superdense coding is a procedure of sending two bits of classical information using only one qubit. An entangled state, a so called Bell state $|\Phi^+\rangle$, is prepared for this experiment. Control over qubit Q0 is then given to Alice, qubit Q2 is given to Bob. Bob manipulates the qubit he was given, depending on what classical information he would like to transmit and sends his qubit to Alice. Alice then perform a Bell state measurement to decode the information.

1. Create the following Bell state with qubits Q0 and Q2.

$$|\Phi^+\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}} \tag{20}$$

   Use only single qubit gates and C-NOT gates. Alice is given control over Q0 and Bob over Q2.

   Solutions: See previous exercise

2. Bob can now manipulate his qubit and sends it to Alice. The following operations are at his disposal

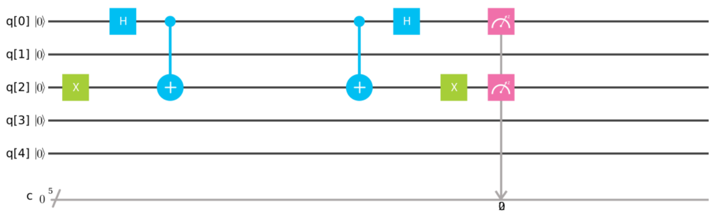
$$\begin{aligned} \text{do nothing}: &\quad \mathbb{I}|\Phi^+\rangle \quad (21)\\ \sigma_x^{(2)}: &\quad \sigma_x^{(2)}|\Phi^+\rangle \quad (22)\\ \sigma_z^{(2)}: &\quad \sigma_z^{(2)}|\Phi^+\rangle \quad (23)\\ i\sigma_y^{(2)}: &\quad \sigma_x^{(2)}\sigma_z^{(2)}|\Phi^+\rangle \quad (24) \end{aligned}$$

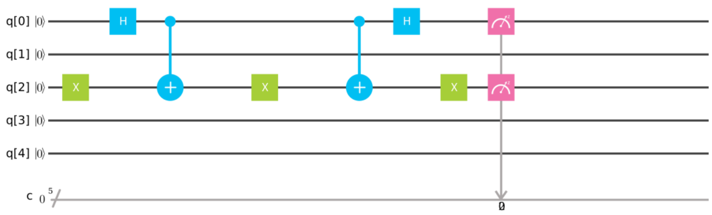
   Determine the output state after each operation. What is the nature of these states?

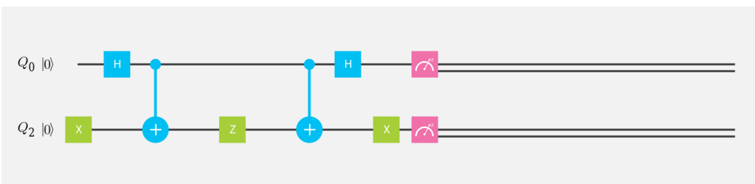
   Solutions: See previous exercise

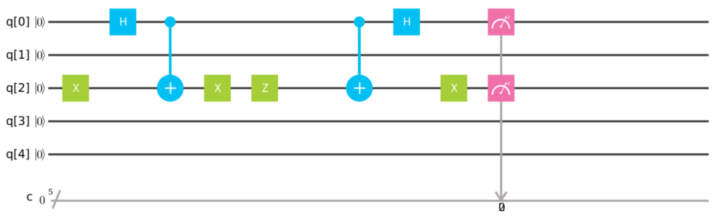
3. Alice performs a measurement in the Bell basis. What is the two bit classical information encoded in all 4 cases?

   Solutions: $00, 01, 10, 11$



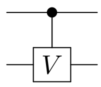






Problem 6: *Controlled phase gate*

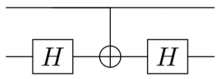
In the previous problem you have created an entangled Bell state using a C-NOT gate. Another basic entangling operation is a conditional Z gate (or conditional phase gate). Such a C-phase gate is described by the following circuit diagram and operator



$$\mathcal{U}_{\mathrm{CZ}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

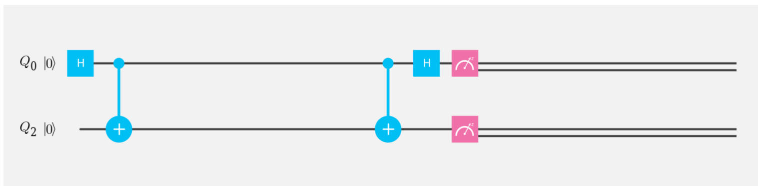
1. Construct a C-phase gate using only single qubit gates and C-NOT.

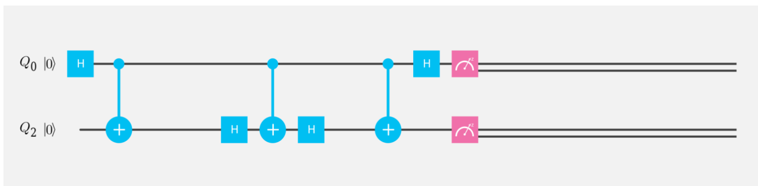
   Solutions:



2. Perform C-phase gate operation on the Bell state $|\Psi^+\rangle$ as prepared in problem 1. Determine the resulting two qubit state and explain why the the operation is called a controlled phase operation. *(Hint: choose an appropriate measurement basis)*

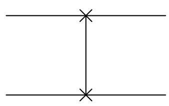
   Solutions:





Problem 7: *SWAP gate*

Many quantum operations such as quantum error correction or reversible logical operations require a permutation of the information stored in a set of qubits. Such permutation can be performed using a so called SWAP gate. A SWAP gate is described by the following circuit diagram and operator ($|00\rangle$,$|01\rangle$,$|10\rangle$,$|11\rangle$ basis). Conduct the following experiment on the real quantum chip ibmqx 4.

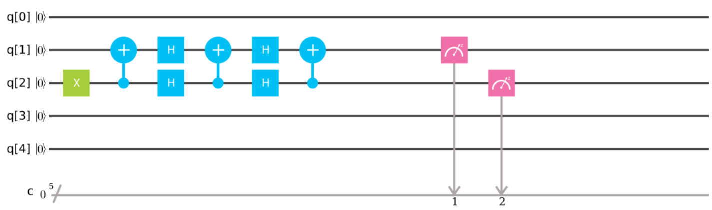


$$\mathcal{U}_{\mathrm{SWAP}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1. Construct a SWAP gate using only single qubit gates and C-NOT.

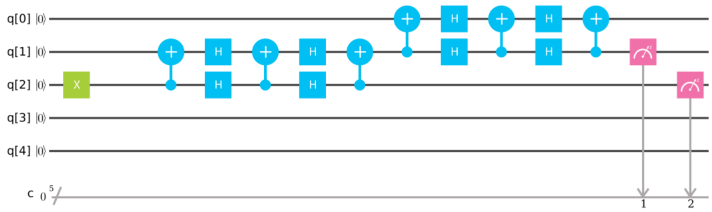
2. Perform a SWAP operation between the qubits Q2 and Q1

   Solutions:



3. Perform a SWAP operation between the qubits Q2 and Q1 followed by Q1 to Q0

   Solutions:



Algorithms

Problem 8: *Grover Search Algorithm*

The Grover Search algorithm is used to search in an unstructured database. It performs the following task: "In an unsorted database (list) of N items there is one satisfying a given condition; retrieve it."[1]. A simple example could be to find a specific card in a deck of cards. The algorithm requires the implementation of a function $f(x)$ which evaluates to 1 for the wanted state x_0, $f(x_0) = 1$ and 0 otherwise ($f(x_{i\neq 0}) = 0$). Classically, $N/2$ items have to be queried on average (N in the worst case), but using the quantum Grover algorithms can reduce the number of queries to approximately $\sqrt{N}$.
The trick is *amplitude amplification*, which boosts the probability of finding the correct state. In short, it inverts the amplitude of all input states about the average. If all states have same amplitude (i.e., probability to be measured/detected), but only the wanted state x_0 has negative amplitude, by one inversion about the average step, all state have positive amplitude with the amplitude of x_0 being above average, and all others below average.
The only thing left is to prepare the state x_0 in a negative amplitude state. This is done by applying the *oracle* operator, $U_f|x\rangle = (-1)^{f(x)}|x\rangle$ to *all* input states. The tricky part is to find the oracle operator. But since in quantum mechanics we can easily create superposition states, appying it to all states at once is simple. This parallelism is at the heart of quantum algorithms.
With this in mind, we can write down the steps required for the Grover search algorithm:

1. State preparation: In this step the qubits are initialized in a superposition state $|\psi_s\rangle$. This is done by applying a Hadamard to each input qubit. Each possible solution state is encoded by a quantum state vector.

2. Application of the *oracle* U_f: $|\psi_s\rangle \to U_f|\psi_s\rangle$ This is the unitary operator that flags the wanted state by inverting its amplitude.

3. Amplitude amplification by inversion using the *diffusion operator* U_s: $U_f|\psi_s\rangle \to U_sU_f|\psi_s\rangle$. In this step all amplitudes are inverted about the mean, which eventually means that the selected state has higher amplitude than the others.

4. Readout. In the ideal case, a single readout is enough to find the correct answer.

[1] Gruza, 'Quantum Computing', McGraw-Hill, 1999)

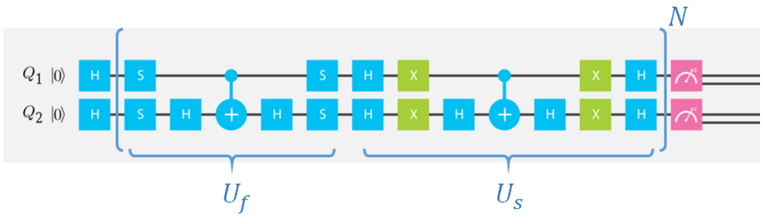


Figure 1: Parts of the Grover search algorithm.

Tasks:

1. Implement the Grover Search algorithm for two-qubits following the tutorial. Run the Quantum Experience with the ideal simulator, the realistic simulator and on the real quantum processor. Describe the differences in the results. Run the real quantum processor also in single-shot mode. Do you always get the right result?

2. Perform the search part of the algorithm two, three and four times. What do you notice in the success probability? (Hint: You have to simplify your algorithm to not exceed the maximal length. Also searching for 11 gives the shortest algorithm.)

3. Verify that before the amplitude amplification the wanted state has inverted amplitude. (Hint: Use measurement pre-rotations to measure along different axes. You have to figure out a set of measurements that allows you to discriminate between the four possible states.)

4. How does the extension of the Grover algorithm to three qubits look like? Write down a realistic version of the algorithm which searches for the 111 state. (Note that this might not be realizable on the Quantum Experience because of too high gate count.)

   Solutions: See QX tutorial